

Lineability and spaceability for the weak form of Peano's theorem and vector-valued sequence spaces

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Abstract

Two new applications of a technique for spaceability are given in this paper. For the first time this technique is used in the investigation of the algebraic genericity property of the weak form of Peano's theorem on the existence of solutions of the ODE $u' = f(u)$ on c_0 . The space of all continuous vector fields f on c_0 is proved to contain a closed \mathfrak{c} -dimensional subspace formed by fields f for which – except for the null field – the weak form of Peano's theorem fails to be true. The second application generalizes known results on the existence of closed \mathfrak{c} -dimensional subspaces inside certain subsets of $\ell_p(X)$ -spaces, $0 < p < \infty$, to the existence of closed subspaces of maximal dimension inside such subsets.

1 Introduction

The notions of lineability and spaceability, as well as their applications, have been heavily studied by many authors in different settings lately, see, e.g., [1, 2, 4, 7, 8, 12, 16] and references therein. The basic task in the field consists in finding linear structures, as large as possible, inside nonempty sets with certain properties. Usually, given a cardinal number μ and a (remarkable) subset A of a topological vector space E , one wishes to show that $A \cup \{0\}$ contains a μ -dimensional subspace of E . According to the usual terminology, A is said to be:

- μ -*lineable* if $A \cup \{0\}$ contains a μ -dimensional subspace of E ;
- μ -*spaceable* if $A \cup \{0\}$ contains a closed μ -dimensional subspace of E ;
- *maximal-spaceable* if it is μ -spaceable with $\mu = \dim E$.

A standard methodology of verifying such properties consists in making a convenient manipulation of a single element of A in order to define an injective linear operator $T: X \rightarrow E$, where X is an infinite dimensional Banach space, such that

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$T(X) \subseteq A \cup \{0\}$. In this case A is $\dim X$ -lineable, and if $\overline{T(X)} \subseteq A \cup \{0\}$, then A is $\dim X$ -spaceable. The starting element of A can be called *mother vector*. The purpose of this paper is to discuss two new applications of the mother vector technique. First, it is explored in a situation it was never applied before (cf. Section 2). Thereafter, the mother vector technique is used to obtain maximal-spaceability in a framework more general than that where \mathfrak{c} -spaceability was obtained in [4] (cf. Section 3). Next we briefly describe the results we prove by means of the mother vector technique.

Throughout this paper, \mathfrak{c} will denote the cardinality of the continuum. Given a Banach space X , we denote by $\mathcal{K}(X)$ the set of all continuous vector fields $f: X \rightarrow X$ for which the weak form of Peano's theorem, concerning the existence of local solutions of

$$u'(t) = f(u(t)),$$

fails to be true. For more details on this current line of research and a historical account, we refer the reader to the recent contribution of Hájek and Johanis [11] and references therein. In Section 2 the mother vector technique is used to prove that $\mathcal{K}(c_0)$ is \mathfrak{c} -spaceable in the space $C(c_0)$ of all continuous vector fields on c_0 endowed with the topology of uniform convergence on bounded subsets. We call this type of property as the algebraic genericity of differential equations in X . The motivation comes from studies on the generic property of differential equations in Banach spaces (see [13]). Our approach to prove this result is based on Dieudonné's construction of vector fields on c_0 failing the classical Peano's theorem (cf. [6]). To the best of our knowledge, this is the first time spaceability is studied in this context.

Our next study concerns maximal-spaceability in the setting of vector-valued sequence spaces. In [4] the mother vector technique was used to prove that if X is a Banach space, then $\ell_p(X) = \bigcup_{q < p} \ell_q(X)$, $0 < p < \infty$, and $c_0(X) = \bigcup_{q > 0} \ell_q(X)$ are \mathfrak{c} -spaceable. In Section 3 we refine the application of the mother vector technique in this context by proving that these sets are actually maximal-spaceable. Furthermore, it is proved that, for $1 \leq p < \infty$, $\bigcap_{p < q} \ell_q(X) = \ell_p(X)$ is maximal-spaceable in the Fréchet space $\bigcap_{p < q} \ell_q(X)$. These results are obtained as particular cases of spaceability results we prove in the more general realm of $(\sum_n X_n)_p$ -spaces. As far as we know, maximal-spaceability with dimension greater than \mathfrak{c} was obtained before only in [8] for sets of non-measurable functions.

2 The weak form of Peano's theorem in c_0

Let X be a Banach space and $f: \mathbb{R} \times X \rightarrow X$ be a continuous vector field on X . The weak form of Peano's theorem states that if X is finite-dimensional, then the ODE

$$u' = f(t, u), \tag{1}$$

has a solution on some open interval I in \mathbb{R} . The study of the failure of Peano's theorem in arbitrary infinite dimensional linear spaces was started by Dieudonné

[6] in 1950. He proved the existence of a continuous vector field $f: c_0 \rightarrow c_0$ such that if $f(t, u) := f(u)$, then the Cauchy-Peano problem associated to (1) has no local solution around the null vector of $\mathbb{R} \times c_0$. Subsequently, counterexamples in ℓ_2 , Hilbert spaces and in nonreflexive Banach spaces were obtained by Yorke [20], Godunov [10] and Cellina [5], respectively. Finally, in 1973 Godunov [9] proved that Peano's theorem holds true in X if and only if X is finite dimensional. Further negative answers were obtained by Astala [3], Shkarin [18, 19], Lobanov [14] and Lobanov and Smolyanov [15] in the setting of locally convex and Fréchet spaces.

Let $C(X)$ denote the linear space of all continuous vector fields on X , which we endow with the linear topology of uniform convergence on bounded sets. From the lineability and spaceability point of view, the following question emerges naturally: How large is the set $\mathcal{K}(X)$ of all fields f in $C(X)$ for which (1) has no local solution? Following the historical development of the subject, it is natural to investigate the case $X = c_0$ first. In this section we give a major step in the solution of the aforementioned question by proving that the set $\mathcal{K}(c_0) \cup \{0\}$ contains a closed \mathfrak{c} -dimensional subspace of $C(c_0)$.

Theorem 2.1. *The set of continuous vector fields on c_0 failing the weak form of Peano's theorem is \mathfrak{c} -spaceable in $C(c_0)$.*

Proof. Let $(e_n)_{n=1}^\infty$ be the canonical unit vectors of sequence spaces and define the vector field $f \in C(c_0)$ by

$$f\left(\sum_{n=1}^\infty x_n e_n\right) = \sum_{n=1}^\infty \left(\sqrt{|x_n|} + \frac{1}{n+1}\right) e_n.$$

By [6] it follows that $f \in \mathcal{K}(c_0)$. Split \mathbb{N} into countably many infinite pairwise disjoint subsets $(\mathbb{N}_i)_{i=1}^\infty$. For every $i \in \mathbb{N}$ set $\mathbb{N}_i = \{i_1 < i_2 < \dots\}$ and define the spreading function $\mathbb{N}_i f: c_0 \rightarrow c_0$ of f over \mathbb{N}_i by

$$\mathbb{N}_i f(x) = \sum_{n=1}^\infty f_{i_n}(x) e_{i_n},$$

where $f_n(x) = \sqrt{|x_n|} + \frac{1}{n+1}$, for all $n \in \mathbb{N}$. Let us see that the map

$$L: \ell_1 \rightarrow C(c_0), \quad L((a_i)_{i=1}^\infty) = \sum_{i=1}^\infty a_i \mathbb{N}_i f,$$

is well defined. Indeed, given $(a_i)_{i=1}^\infty \in \ell_1$ and $x \in c_0$, for every $m \in \mathbb{N}$ we have (the sup norm on c_0 is simply denoted by $\|\cdot\|$)

$$\left\| \sum_{i=1}^m a_i \mathbb{N}_i f(x) \right\| \leq \sum_{i=1}^m |a_i| \cdot \|\mathbb{N}_i f(x)\| \leq \sum_{i=1}^m |a_i| \cdot \|f(x)\| = \|f(x)\| \left(\sum_{i=1}^m |a_i| \right).$$

Making $m \rightarrow \infty$ we conclude that $\sum_{i=1}^{\infty} a_i \mathbb{N}_i f(x) \in c_0$. Now let us show that $L((a_i)_{i=1}^{\infty}) \in C(c_0)$. For all $x, y \in c_0$ and $m \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{i=1}^m a_i \mathbb{N}_i f(x) - \sum_{i=1}^m a_i \mathbb{N}_i f(y) \right\| &\leq \sum_{i=1}^m |a_i| \cdot \|\mathbb{N}_i f(x) - \mathbb{N}_i f(y)\| \\ &\leq \|f(x) - f(y)\| \left(\sum_{i=1}^m |a_i| \right). \end{aligned}$$

Then by taking the limit as $m \rightarrow \infty$, the continuity of $\sum_{i=1}^{\infty} a_i \mathbb{N}_i f$ follows from the continuity of f . Since L is well defined, its linearity and injectivity are clear, so the range space $L(\ell_1)$ is algebraically isomorphic to ℓ_1 . We claim that

$$\overline{L(\ell_1)} \subseteq \mathcal{K}(c_0) \cup \{0\}. \quad (2)$$

Let $h = (h_i)_{i=1}^{\infty} \in \overline{L(\ell_1)}$ be arbitrary. We may assume that $h \neq 0$, so there is $r \in \mathbb{N}$ such that $h_r \neq 0$. Using the decomposition $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$ there are (unique) $m, s \in \mathbb{N}$ such that $e_{ms} = e_r$. Let $(x_k)_{k=1}^{\infty} = ((a_i^k)_{i=1}^{\infty})_{k=1}^{\infty}$ be a sequence in ℓ_1 so that

$$L(x_k) = \sum_{j=1}^{\infty} a_j^k \mathbb{N}_j f \xrightarrow{k \rightarrow \infty} h \text{ in } C(c_0).$$

Letting $L_n(x_k)$ denote the n -th coordinate of $L(x_k)$, for each $N \in \mathbb{N}$ we have that

$$h_n = \lim_{k \rightarrow \infty} L_n(x_k)$$

uniformly in the ball $B_{c_0}(N) := \{x \in c_0 : \|x\| \leq N\}$. Since $L_{i_j}(x_k) = a_i^k f_{i_j}$ for all $i, j \in \mathbb{N}$, it follows that

$$a_i^k f_{i_j}(x) = L_{i_j}(x_k)(x) \xrightarrow{k \rightarrow \infty} h_{i_j}(x)$$

for each $x \in B_{c_0}(N)$. In particular,

$$a_m^k f_{m_j}(x) = L_{m_j}(x_k)(x) \xrightarrow{k \rightarrow \infty} h_{m_j}(x) \quad (3)$$

for every j and every $x \in B_{c_0}(N)$; and making $j = s$ we get

$$a_m^k f_r(x) = a_m^k f_{m_s}(x) \xrightarrow{k \rightarrow \infty} h_{m_s}(x) = h_r(x) \quad (4)$$

for each $x \in B_{c_0}(N)$. Choosing $x_0 \in c_0$ such that $h_r(x_0) \neq 0$ and $N_0 \in \mathbb{N}$ such that $x_0 \in B_{c_0}(N_0)$ it follows that

$$a_r := \lim_{k \rightarrow \infty} a_m^k = \frac{h_r(x_0)}{f_r(x_0)} \neq 0.$$

Thus (4) implies that $h_r(x) = a_r f_r(x)$ for all $x \in B_{c_0}(N)$. As N is arbitrary, we have $h_r(x) = a_r f_r(x)$ for every $x \in c_0$. Since for every $j, k \in \mathbb{N}$ the m_j -th coordinate of $L(x_k)$ is $a_m^k f_{m_j}$, by (3) we have that $h_{m_j}(x) = a_r f_{m_j}(x)$, for all $j \in \mathbb{N}$.

Now we are ready to prove that $h \in \mathcal{K}(c_0)$. We proceed by contradiction using the ODE approach from [6]. Assume that $u(t) = (u_n(t))_{n=1}^\infty$ is a solution of (1) on some interval $I \subset \mathbb{R}$. Fix any $a \in I$ and write $b = u(a) = (b_i)_{i=1}^\infty$. In this case we have

$$u'_{m_j}(t) = h_{m_j}(u(t)) = a_r f_{m_j}(u(t)) = a_r \left(\sqrt{|u_{m_j}(t)|} + \frac{1}{m_j + 1} \right),$$

and $u_{m_j}(a) = b_{m_j}$ for all $j \in \mathbb{N}$ and $t \in I$. In summary, each u_{m_j} is a solution of the Cauchy problem

$$u'_{m_j}(t) = a_r \left(\sqrt{|u_{m_j}(t)|} + \frac{1}{m_j + 1} \right), \quad u_{m_j}(a) = b_{m_j}, \quad (5)$$

for all $j \in \mathbb{N}$ and all $t \in I$, and hence for all $t \in \mathbb{R}$. Let us recall the original argument of Dieudonné [6]: if $\alpha, \beta \in \mathbb{R}$ and $\gamma > 0$ then

$$\int_\alpha^\beta \frac{dx}{\sqrt{|x|} + \gamma} \leq 2(\sqrt{|\alpha|} + \sqrt{|\beta|}).$$

Thus if $u'(t) = \lambda \left(\sqrt{|u(t)|} + \gamma \right)$ with $\gamma, \lambda > 0$, $t \geq t_0$ and $u(t_0) = y_0$, then

$$t - t_0 = \int_{t_0}^t \frac{u'(s) ds}{\lambda \left(\sqrt{|u(s)|} + \gamma \right)} = \frac{1}{\lambda} \int_{u(t_0)}^{u(t)} \frac{dx}{\sqrt{|x|} + \gamma} \leq \frac{2}{\lambda} \left(\sqrt{|u(t)|} + \sqrt{|u(t_0)|} \right).$$

In view of (5), if $a_r > 0$ then

$$0 < \frac{a_r(t - a)}{2} \leq \sqrt{|u_{m_j}(t)|} + \sqrt{|u_{m_j}(a)|}$$

for all $t > a$ and all $j \in \mathbb{N}$. This contradiction – remember that $(u_{m_j}(t))_{j \in \mathbb{N}} \in c_0$ for $t \in I$ – shows that $h \in \mathcal{K}(c_0)$. If $a_r < 0$, we can define $v_{m_j}(t) = u_{m_j}(-t)$ for all $t \in \mathbb{R}$ and $j \in \mathbb{N}$. Applying once more Dieudonné's argument we get

$$0 < \frac{-a_r(t - a)}{2} \leq \sqrt{|u_{m_j}(-t)|} + \sqrt{|u_{m_j}(-a)|}$$

for all $t > a$ and all $j \in \mathbb{N}$; which is impossible since $(u_{m_j}(-t))_{j \in \mathbb{N}} \in c_0$ for $t \in I$. Hence $h \in \mathcal{K}(c_0)$.

So (2) is established, proving that $\overline{L(\ell_1)}$ is a closed \mathfrak{c} -dimensional subspace of $C(c_0)$ contained in $\mathcal{K}(c_0) \cup \{0\}$. \square

3 Vector-valued sequence spaces

Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Given $0 < p < \infty$, by $(\sum_n X_n)_p$ we mean the vector space of all sequences $(x_n)_{n=1}^\infty$ such that $x_n \in X_n$ for every n and

$$\|(x_n)_{n=1}^\infty\|_p := \left(\sum_{n=1}^\infty \|x_n\|_{X_n}^p \right)^{1/p} < \infty.$$

It is well known that $((\sum_n X_n)_p, \|\cdot\|_p)$ is a Banach (p -Banach if $0 < p < 1$) space. Making the obvious modification for $p = 0$ we get the Banach space $(\sum_n X_n)_0$ of norm null sequences with the sup norm. In this fashion,

$$(\sum_n X_n)_p^- := \bigcup_{0 < q < p} (\sum_n X_n)_q$$

can be regarded as a subspace of $(\sum_n X_n)_p$ and $\bigcup_{p>0} (\sum_n X_n)_p$ can be regarded as a subspace of $(\sum_n X_n)_0$.

For a Banach space X , the usual X -valued sequence spaces $\ell_p(X)$ and $c_0(X)$, as well as their corresponding subspaces $\ell_p^-(X) := \bigcup_{0 < q < p} \ell_q(X)$ and $\bigcup_{p>0} \ell_p(X)$, are recovered putting $X_n = X$ for every n . In [4] it is proved that $\ell_p(X) - \ell_p^-(X)$ and $c_0(X) - \bigcup_{p>0} \ell_p(X)$ are \mathfrak{c} -spaceable. In this section we shall prove that these sets are actually maximal-spaceable (cf. Corollary 3.5). This information will be obtained as a particular case of spaceability results in the more general realm of $(\sum_n X_n)_p$ -spaces.

Definition 3.1. Given a Banach space X , a family $(X_i)_{i \in I}$ of Banach spaces is said to *contain isomorphs of X uniformly* if there are $\delta > 0$ and a family of isomorphisms into $R_i: X \rightarrow X_i$ such that $\min\{\|R_i\|, \|R_i^{-1}\|\} \leq \delta$ for every $i \in I$.

Theorem 3.2. *Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces that contains a subsequence containing isomorphs of the infinite dimensional Banach space X uniformly. Then:*

- (a) $(\sum_n X_n)_p - (\sum_n X_n)_p^-$ is $\dim X$ -spaceable for every $0 < p < \infty$.
- (b) $(\sum_n X_n)_0 - \bigcup_{p>0} (\sum_n X_n)_p$ is $\dim X$ -spaceable.

Proof. It is plain that we can assume, without loss of generality, that the sequence $(X_n)_{n=1}^\infty$ contain isomorphs of X uniformly. So there are $\delta > 0$ and isomorphisms into $R_n: X \rightarrow X_n$ such that $\|R_n\| \leq \delta$ and $\|R_n^{-1}\| \leq \delta$ for every $n \in \mathbb{N}$.

- (a) Let $\xi = (\xi_j)_{j=1}^\infty \in \ell_p - \bigcup_{0 < q < p} \ell_q$. Split \mathbb{N} into countably many infinite pairwise disjoint subsets $(\mathbb{N}_i)_{i=1}^\infty$. For every $i \in \mathbb{N}$ set $\mathbb{N}_i = \{i_1 < i_2 < \dots\}$ and, denoting by $(e_n)_{n=1}^\infty$ the canonical unit vectors of sequence spaces, define

$$y_i = \sum_{j=1}^\infty \xi_j e_{i_j} \in \mathbb{K}^\mathbb{N}.$$

Since $\|y_i\|_r = \|\xi\|_r$ for every $r > 0$, we have that $y_i \in \ell_p - \bigcup_{0 < q < p} \ell_q$, for every i . For $x = (x_n)_{n=1}^\infty \in \mathbb{K}^\mathbb{N}$ and $w \in X$ we write

$$x \otimes w := (x_n R_n(w))_{n=1}^\infty \in \prod_{n=1}^\infty X_n.$$

It is clear that, for all $w, w_1, w_2 \in X$ and $\lambda \in \mathbb{K}$,

$$x \otimes (w_1 + w_2) = x \otimes w_1 + x \otimes w_2 \quad \text{and} \quad \lambda(x \otimes w) = (\lambda x) \otimes w = x \otimes (\lambda x), \quad (6)$$

what justifies the use of the symbol \otimes . Define $\tilde{s} = 1$ if $p \geq 1$ and $\tilde{s} = p$ if $0 < p < 1$. Let $(w_j)_{j=1}^\infty \in \ell_{\tilde{s}}(X)$ be given. As $y_j \otimes w_j \in \prod_{n=1}^\infty X_n$ for every $j \in \mathbb{N}$, and

$$\begin{aligned} \|y_j \otimes w_j\|_p &= \left(\sum_{k=1}^\infty |\xi_k|^p \cdot \|R_{j_k}(w_j)\|_{X_{j_k}}^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^\infty |\xi_k|^p \cdot \|R_{j_k}\|^p \cdot \|w_j\|_X^p \right)^{\frac{1}{p}} \\ &\leq \delta \|w_j\|_X \cdot \|\xi\|_p < \infty, \end{aligned}$$

each $y_j \otimes w_j \in (\sum_n X_n)_p$. Moreover,

$$\sum_{j=1}^\infty \|y_j \otimes w_j\|_p^{\tilde{s}} \leq \sum_{j=1}^\infty (\delta \|w_j\|_X \cdot \|\xi\|_p)^{\tilde{s}} = \delta^{\tilde{s}} \|\xi\|_p^{\tilde{s}} \cdot \|(w_j)_{j=1}^\infty\|_{\tilde{s}}^{\tilde{s}} < \infty.$$

Thus $\sum_{j=1}^\infty \|y_j \otimes w_j\|_p < \infty$ if $p \geq 1$ and $\sum_{j=1}^\infty \|y_j \otimes w_j\|_p^p < \infty$ if $0 < p < 1$. Hence the series $\sum_{j=1}^\infty y_j \otimes w_j$ converges in $(\sum_n X_n)_p$ and the operator

$$T: \ell_{\tilde{s}}(X) \longrightarrow (\sum_n X_n)_p, \quad T((w_j)_{j=1}^\infty) = \sum_{j=1}^\infty y_j \otimes w_j,$$

is well defined. From (6) it follows easily that T is linear, and from the fact that the sequences $(y_j)_{j=1}^\infty$ are disjointly supported it follows that T is injective. Thus $\overline{T(\ell_{\tilde{s}}(X))}$ is a closed infinite dimensional subspace of $(\sum_n X_n)_p$ and

$$\dim \overline{T(\ell_{\tilde{s}}(X))} = \dim \ell_{\tilde{s}}(X) = \dim X.$$

Now we just have to show that

$$\overline{T(\ell_{\tilde{s}}(X))} - \{0\} \subseteq (\sum_n X_n)_p - \bigcup_{0 < q < p} (\sum_n X_n)_q.$$

Let $z = (z_n)_{n=1}^\infty \in \overline{T(\ell_{\tilde{s}}(X))}$, $z \neq 0$. There are sequences $(w_i^{(k)})_{i=1}^\infty \in \ell_{\tilde{s}}(X)$, $k \in \mathbb{N}$, such that $z = \lim_{k \rightarrow \infty} T\left(\left(w_i^{(k)}\right)_{i=1}^\infty\right)$ in $(\sum_n X_n)_p$. Note that, for each $k \in \mathbb{N}$,

$$T\left(\left(w_i^{(k)}\right)_{i=1}^\infty\right) = \sum_{i=1}^\infty y_i \otimes w_i^{(k)} = \sum_{i=1}^\infty \left(\sum_{j=1}^\infty \xi_j e_{i_j} \right) \otimes w_i^{(k)} = \sum_{i=1}^\infty \sum_{j=1}^\infty \xi_j e_{i_j} \otimes w_i^{(k)},$$

what means that, for every $i, j \in \mathbb{N}$, the i -th coordinate of $T\left(\left(w_i^{(k)}\right)_{i=1}^{\infty}\right)$ is $\xi_j R_{ij}(w_i^{(k)})$. Fix $r \in \mathbb{N}$ such that $z_r \neq 0$. Since $\mathbb{N} = \bigcup_{j=1}^{\infty} \mathbb{N}_j$, there are (unique) $m, t \in \mathbb{N}$ such that $m_t = r$. Thus, for each $k \in \mathbb{N}$, the r -th coordinate of $T\left(\left(w_i^{(k)}\right)_{i=1}^{\infty}\right)$ is $\xi_t R_r(w_m^{(k)})$. Since convergence in $(\sum_n X_n)_p$ implies coordinatewise convergence, we have $z_r = \lim_{k \rightarrow \infty} \xi_t R_r(w_m^{(k)})$. From $z_r \neq 0$ it follows that $\xi_t \neq 0$ and

$$z_r = \lim_{k \rightarrow \infty} \xi_t R_r(w_m^{(k)}) = \xi_t \cdot \lim_{k \rightarrow \infty} R_r(w_m^{(k)}) \in \overline{R_r(X)} = R_r(X).$$

Hence $R_r^{-1}(z_r) = \xi_t \cdot \lim_{k \rightarrow \infty} w_m^{(k)}$. Call

$$\alpha_m := \lim_{k \rightarrow \infty} w_m^{(k)} = \frac{R_r^{-1}(z_r)}{\xi_t} \neq 0.$$

For $j, k \in \mathbb{N}$, the m_j -th coordinate of $T\left(\left(w_i^{(k)}\right)_{i=1}^{\infty}\right)$ is $\xi_j R_{m_j}(w_m^{(k)})$. On the one hand,

$$\lim_{k \rightarrow \infty} \xi_j R_{m_j}(w_m^{(k)}) = \xi_j \cdot \lim_{k \rightarrow \infty} R_{m_j}(w_m^{(k)}) = \xi_j R_{m_j}\left(\lim_{k \rightarrow \infty} w_m^{(k)}\right) = \xi_j R_{m_j}(\alpha_m)$$

for every $j \in \mathbb{N}$. On the other hand, coordinatewise convergence gives

$$\lim_{k \rightarrow \infty} \xi_j R_{m_j}(w_m^{(k)}) = z_{m_j},$$

so $z_{m_j} = \xi_j R_{m_j}(\alpha_m)$ for each $j \in \mathbb{N}$.

Finally, for all $0 < q < p$,

$$\begin{aligned} \|z\|_q^q &= \sum_{n=1}^{\infty} \|z_n\|_{X_n}^q \geq \sum_{j=1}^{\infty} \|z_{m_j}\|_{X_{m_j}}^q = \sum_{j=1}^{\infty} |\xi_j|^q \cdot \|R_{m_j}(\alpha_m)\|_{X_{m_j}}^q \\ &\geq \sum_{j=1}^{\infty} |\xi_j|^q \cdot \frac{1}{\|R_{m_j}^{-1}\|^q} \cdot \|\alpha_m\|_X^q \geq \frac{1}{\delta^q} \|\alpha_m\|_X^q \cdot \|\xi\|_q^q = \infty, \end{aligned}$$

proving that $z \notin \bigcup_{0 < q < p} (\sum_n X_n)_q$.

(b) Start with a sequence $\xi = (\xi_j) \in c_0 - \bigcup_{p>0} \ell_p$ and proceed as before to define the operator

$$T: \ell_1(X) \longrightarrow (\sum_n X_n)_0, \quad T\left((w_j)_{j=1}^{\infty}\right) = \sum_{j=1}^{\infty} y_j \otimes w_j.$$

The well-definiteness of T is even easier in this case. Again T is linear, injective and the same steps of the proof of (a) show that

$$\overline{T(\ell_1(X))} - \{0\} \subseteq (\sum_n X_n)_0 - \bigcup_{p>0} (\sum_n X_n)_p.$$

As $\dim \overline{T(\ell_1(X))} = \dim X$, the proof is complete. \square

Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces and $1 \leq p < +\infty$. We define

$$(\sum_n X_n)_p^+ := \bigcap_{q>p} (\sum_n X_n)_q = \bigcap_{k \in \mathbb{N}} (\sum_n X_n)_{p_k},$$

where $(p_k)_{k=1}^\infty$ is any decreasing sequence converging to p , endowed with the locally convex topology τ generated by the family of norms

$$\|(x_n)_{n=1}^\infty\|_q = \left(\sum_{n=1}^\infty \|x_n\|_{X_n}^q \right)^{\frac{1}{q}}, \quad q > p.$$

This locally convex topology τ is clearly generated by the countably family of norms

$$\|(x_n)_{n=1}^\infty\|_{p_k} = \left(\sum_{n=1}^\infty \|x_n\|_{X_n}^{p_k} \right)^{\frac{1}{p_k}}, \quad k \in \mathbb{N},$$

so $((\sum_n X_n)_p^+, \tau)$ is metrizable. The completeness can be proved similarly to the case of $(\sum_n X_n)_p$. Alternatively, note that τ is the projective limit topology defined by the inclusions $(\sum_n X_n)_p^+ \hookrightarrow (\sum_n X_n)_q$, $q > p$. So it is complete as the projective limit of complete Hausdorff spaces. In summary $((\sum_n X_n)_p^+, \tau)$ is a Fréchet space.

If $X_n = X$ for every n , we write $\ell_p(X)^+$. In particular, for $X = \mathbb{K}$, $\ell_p^+ := \ell_p^+(\mathbb{K})$ recovers the space l^{p+} introduced by Metafune and Moscatelli [17].

Remark 3.3. It is easy to see that if $X_n \neq \{0\}$ for every n , then the inclusions

$$(\sum_n X_n)_p \subseteq (\sum_n X_n)_p^+ \subseteq (\sum_n X_n)_q$$

are strict whenever $1 \leq p < q$.

Theorem 3.4. *Let $(X_n)_{n=1}^\infty$ be a sequence of Banach spaces that contains a subsequence containing isomorphs of the infinite dimensional Banach space X uniformly. Then $(\sum_n X_n)_p^+ - (\sum_n X_n)_p$ is $\dim X$ -spaceable.*

Proof. The proof goes along the same steps of the proof of Theorem 3.2. We sketch the argument highlighting the differences the locally convex topology of $(\sum_n X_n)_p^+$ arises. In this proof, *as before* means *as in the proof of Theorem 3.2*.

Let $\xi = (\xi_j) \in \ell_p^+ - \ell_p$. Write $\mathbb{N} = \bigcup_{i=1}^\infty \mathbb{N}_i$, $\mathbb{N}_i = \{i_1 < i_2 < \dots\}$, and define $y_j \in \mathbb{K}^\mathbb{N}$, $j \in \mathbb{N}$, as before. Since $\|y_i\|_r = \|\xi\|_r$ for every $r > 0$, we have that $y_i \in \ell_p^+ - \ell_p$, for every i . Given $(w_j)_{j=1}^\infty \in \ell_1(X)$, let us show that the series $\sum_{j=1}^\infty y_j \otimes w_j$ converges in $(\sum_n X_n)_p^+$, where $y_j \otimes w_j$ is defined as before. Write $s_n = \sum_{j=1}^n y_j \otimes w_j$, $n \in \mathbb{N}$. For a fixed $q > p$, the same computations we performed before show that each $y_j \otimes w_j \in (\sum_n X_n)_q$ and

$$\sum_{j=1}^n \|y_j \otimes w_j\|_q \leq \delta \|\xi\|_q \cdot \|(w_j)_{j=1}^\infty\|_1$$

for every n . As $(\sum_n X_n)_q$ is a Banach space, there is $S_q \in (\sum_n X_n)_q$ such that $S_q = \lim_{n \rightarrow \infty} s_n$ in $(\sum_n X_n)_q$. If $q, q' > p$, say $q \leq q'$, then $S_q \in (\sum_n X_n)_{q'}$ and

$$\|s_n - S_q\|_{q'} \leq \|s_n - S_q\|_q \rightarrow 0,$$

showing that $S_q = \lim_{n \rightarrow \infty} s_n$ in $(\sum_n X_n)_{q'}$, therefore $S_q = S_{q'}$. This shows that S_q does not depend on q , so there is $S \in (\sum_n X_n)_q$ such that $s_n \rightarrow S$ in $(\sum_n X_n)_q$ for every $q > p$. Hence $S \in (\sum_n X_n)_p^+$ and $s_n \rightarrow S$ in the topology of $(\sum_n X_n)_p^+$. In other words, $\sum_{j=1}^{\infty} y_j \otimes w_j \in (\sum_n X_n)_p^+$ and the operator

$$T: \ell_1(X) \rightarrow (\sum_n X_n)_p^+ \quad , \quad T((w_j)_{j=1}^{\infty}) = \sum_{j=1}^{\infty} y_j \otimes w_j$$

is then well defined. As before, T is linear and injective. Thus $\overline{T(\ell_1(X))}$ is a closed $\dim X$ -dimensional subspace of $(\sum_n X_n)_p^+$. Now we just have to show that

$$\overline{T(\ell_1(X))} - \{0\} \subseteq (\sum_n X_n)_p^+ - (\sum_n X_n)_p.$$

Let $z = (z_n)_{n=1}^{\infty} \in \overline{T(\ell_1(X))}$, $z \neq 0$. There are sequences $w_k = (w_i^{(k)})_{i=1}^{\infty} \in \ell_1(X)$, $k \in \mathbb{N}$, such that $z = \lim_{k \rightarrow \infty} T((w_i^{(k)})_{i=1}^{\infty})$ in $(\sum_n X_n)_p^+$. Since the topology of $(\sum_n X_n)_p^+$ is generated by the norms $\|\cdot\|_q$, $q > p$, it follows that

$$\lim_{k \rightarrow \infty} \left\| T((w_i^{(k)})_{i=1}^{\infty}) - z \right\|_q = 0 \text{ for every } q > p.$$

Fix $q > p$ and use coordinatewise convergence in $(\sum_n X_n)_q$ as before to conclude that $z \notin (\sum_n X_n)_p$. \square

Making $X_n = X$ for every n in Theorems 3.2 and 3.4, since $\dim \ell_p(X) = \dim c_0(X) = \dim \ell_p(X)^+ = \dim X$, we get:

Corollary 3.5. *Let X be an infinite dimensional Banach space X . Then:*

- (a) $\ell_p(X) - \ell_p(X)^-$ is maximal-spaceable for every $0 < p < \infty$.
- (b) $c_0(X) - \bigcup_{p>0} \ell_p(X)$ is maximal-spaceable.
- (c) $\ell_p(X)^+ - \ell_p(X)$ is maximal-spaceable for every $1 \leq p < \infty$.

Of course Theorems 3.2 and 3.4 apply to many other interesting situations, but we refrain from going into details.

Remark 3.6. Note that in the proofs of Theorems 3.2 and 3.4 we start with a vector ξ belonging to a set other than the one we prove to be spaceable. It is this variation of the mother vector technique that allows us obtain maximal-spaceability rather than \mathfrak{c} -spaceability in Corollary 3.5.

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